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Statistical Physics IV : Itô Calculus and Black Scholes

Blackboard derivation for Black-Scholes part

Contents

1. Introduction : Options

A European Option is a contract giving its holder the right (but not the obligation) to buy ("Call option") or sell ("put" option) an asset (the underlying) at a future time T at the predetermined delivery price k (the "strike"). The seller of the option has the obligation to deliver the underlying to the buyer if he decides to exercise the option.

Option are often used as an insurance (hedging) or speculative tool (e.g Airliners or currency hedge), thus it is important to find a method to price them.

The asset we have the right to buy or sell is called the *underlying* and can be almost any financial instrument :

- A Stock (Nestlé SWX:NESN)
- A currency pair (EUR/CHF)
- A metal (gold)
- A commodity (crude oil)
- (A bond (Swiss Nat. Bank Bills))

The payoff (gains) of a call option is mathematically defined as $\Phi(S_T) \equiv (S_T - k)^+ = \max(S_T - k, 0)$ as we consider that a rational investor would never exercise the option if it is not profitable for him.

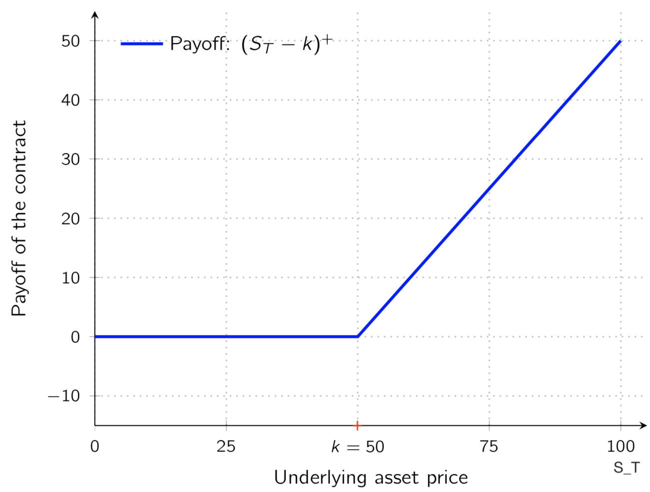

Figure 1: Profit and Loss of a Long Call EU Option with strike $k = 50$ and initial price c_0

The price to enter an option contract depends on the current price of the underlying $S(t)$, the time to maturity and the strike price k. It can be hard to compute as it require to predict the price of the underlying in advance. This is exactly the goal of Black and Scholes' model.

2. Price of an European Option

Consider a stock S (or any other type of asset), of price $S(t)$ such that

$$\mathrm{d}S(t) = S(t)\mu\mathrm{d}t + S(t)\sigma\mathrm{d}W(t) \tag{1}$$

i.e it follows a Geometric Brownian Motion. In other word, the *return* on S (or relative evolution), $\frac{\mathrm{d}S(t)}{S(t)}$, follows a linear growth (the "drift") plus a certain volatility component (the "noise").

Geometric Brownian Motion : Solution of the SDE

Rewriting the dynamics (1) as

$$\frac{\mathrm{d}S_t}{S_t} = \mu\mathrm{d}t + \sigma\mathrm{d}W_t$$

one recognize the differential of a log, hence let's define $Y_t = \log(S_t)$, which is also a stochastic process. Ito's lemma gives :

$$\begin{aligned}\mathrm{d}Y_t &= \frac{\partial Y_t}{\partial S}\mathrm{d}S_t + \frac{1}{2}\frac{\partial^2 Y_t}{\partial S^2}\mathrm{d}S_t \\ &= \frac{\mathrm{d}S_t}{S_t} - \frac{1}{2}\frac{\mathrm{d}S_t^2}{S_t^2}\end{aligned}$$

Using the fact that $\mathrm{d}t^2 = 0$, $\mathrm{d}W_t^2 = \mathrm{d}t$ and $\mathrm{d}t\mathrm{d}W_t \sim \mathrm{d}W_t^3 = 0$, we have

$$\begin{aligned}\mathrm{d}Y_t &= \mu\mathrm{d}t + \sigma\mathrm{d}W_t - \frac{1}{2}\sigma^2\mathrm{d}t \\ &= (\mu - \frac{1}{2}\sigma^2)\mathrm{d}t + \sigma S_t\mathrm{d}W_t\end{aligned}$$

As μ and σ are constants[1], one can integrate Y_t between $t_0 = 0$ and t :

$$Y_t = (\mu - \frac{1}{2}\sigma^2)t + \sigma W_t \tag{2}$$

such that

$$S_t = e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t} \tag{3}$$

This right to buy or sell indeed comes at a given price F, which depends on the current time t and the current underlying price $S(t)$ (not its history) hence $F = F(S(t), t)$.

Applying Ito's lemma yield

$$\mathrm{d}F(S(t),t) = \frac{\partial F}{\partial t}\mathrm{d}t + \frac{\partial F}{\partial S}\underbrace{(\mu S_t\mathrm{d}t + \sigma S_t\mathrm{d}W_t)}_{\mathrm{d}S_t} + \frac{1}{2}\frac{\partial^2 F}{\partial S^2}\underbrace{\sigma^2 S_t^2\mathrm{d}t}_{\mathrm{d}S_t^2}$$

Collecting all the $\mathrm{d}t$ (the deterministic or riskless[2] part) and $\mathrm{d}W_t$ (stochastic or risky part) terms

$$\mathrm{d}F(S(t),t) = \left(\frac{\partial F}{\partial t} + \mu S(t)\frac{\partial F}{\partial S} + \frac{1}{2}\sigma^2 S(t)^2\frac{\partial^2 F}{\partial S^2}\right)\mathrm{d}t + \sigma S(t)\frac{\partial F}{\partial S}\mathrm{d}W(t) \tag{4}$$

This is the dynamics of our option's price.

[1] That's an assumption of the model

[2] The deterministic part is said to be riskless as it can be integrated exactly : knowing the value of the process at t_0 is enough to know the value at any time after t_0.

3. Hedged (riskless) Portfolio

The dynamics of F includes a deterministic part plus a stochastic one which brings in risk. By combining in a specific way the option and its underlying stock it is possible to remove this risky component.

One way is to buy one option and compensate its risk by selling Δ shares of stock S :

$$V(S,t) = F(S,t) - \Delta \cdot S(t). \tag{5}$$

Which proportion of stock Δ is needed to remove risk ? We want the portfolio to be insensible to a variation of $S(t)$ i.e

$$0 \equiv \frac{\partial V}{\partial S} = \frac{\partial F}{\partial S} - \Delta \implies \Delta = \frac{\partial F}{\partial S} \tag{6}$$

One can check that the dynamics of $V(S,t)$ is deterministic and hence riskless :

$$dV(s,t) = dF(s,t) - \frac{\partial F}{\partial s} dS(t) \tag{7}$$

$$= \left(\frac{\partial F}{\partial t} + \mu s \frac{\partial F}{\partial s} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 F}{\partial s^2}\right) \mathrm{d}t + \sigma s \frac{\partial F}{\partial s} \mathrm{d}W(t) - \left(\mu s \frac{\partial F}{\partial s} \mathrm{d}t + \sigma s \frac{\partial F}{\partial s} \mathrm{d}W(t)\right) \tag{8}$$

$$= \left(\frac{\partial F}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 F}{\partial s^2}\right) \mathrm{d}t \tag{9}$$

which indeed has no stochastic component : the uncertainty on the option is exactly balanced by the uncertainty on the stock. This technique, called "Delta Hedging" is slightly different than Black and Scholes initial argument. See [1] for more details on this point.

4. Putting all together : Black Scholes PDE

As the portfolio is riskless, the no arbitrage principle requires that it has the same return than a risk free bank account with return r i.e its dynamics should be given by

$$\mathrm{d}V(s,t) = rV(s,t)\mathrm{d}t \tag{10}$$

Combining eqs (9) and (10) gives

$$\mathrm{d}V(s,t) = rF(s,t) - rs\frac{\partial F}{\partial s} = \left(\frac{\partial F}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 F}{\partial s^2}\right) \mathrm{d}t$$

Rearranging the term yields the famous Black-Scholes PDE

$$\frac{\partial F}{\partial t} + rs\frac{\partial F}{\partial s} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 F}{\partial s^2} - rF(s,t) = 0, \tag{11}$$

$$F(T,s) = \Phi(s) \tag{12}$$

One remarkable fact is that the drift constant (or mean return) μ does not appear in eq (11). In fact the price of the option is not absolute but **relative** to the underlying price.

Also if $r = 0$, (11) becomes a diffusion equation with $D = \frac{1}{2}\sigma^2 S^2$ (true for the log)

5. Solving the equation (sketch)

One way to solve the PDE is to cast it into a backward Fokker-Planck equation for the transition probability $P(x,T|s,t)$:

$$\frac{\partial P(x,T|s,t)}{\partial t} + rs\frac{\partial P(x,T|s,t)}{\partial s} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 P(x,T|s,t)}{\partial s^2} = 0, \tag{13}$$

$$P(x,T|s,T) = \delta(x-s) \tag{14}$$

in which case the solution reads :

$$F(s,t) = e^{-r(T-t)} \int \mathrm{d}x P(x,T|s,t) F(x,T) \tag{15}$$

where $F(x,T) \equiv \Phi(x)$ is the terminal value. For a call option (the option to *buy*),

$$\Phi(x) = (x-k)^+ = (x-k) \cdot \Theta(x-k)$$

such that integral solution simplifies :

$$F(s,t) = e^{-r(T-t)} \int_k^\infty \mathrm{d}x (x-k) P(x,T|s,t). \tag{16}$$

The last step is to find an expression for the transition probabilities. This is given by the Fokker-Planck theory (see [2] ch. 4.3), which links the PDE (13) to the following SDE :

$$\mathrm{d}x(t) = rx(t)\mathrm{d}t + \sigma x(t) \mathrm{d}W_t \tag{17}$$

i.e. the SDE of a geometric brownian motion. $x(t)$ is log-normally distributed hence we have that $\log(x)$ follows a gaussian distribution. Knowing that one can derive an expression for the transition probabilities $P(x,T|s,t)$. Then the integral expression for $F(s,t)$ can be evaluated in term of the cumulative density function of the Gaussian

$$\phi(x) \equiv \int_x^\infty \mathrm{d}x \, e^{-\frac{1}{2}x^2}$$

Eventually, in the case of a call option $\Phi(S_T) = (S_T - k)^+$ the solution of Black-Scholes' PDE is

$$F(S_t,t) = S_t \, \phi(d_1) - ke^{-r(T-t)} \, \phi(d_2) \tag{18}$$

where

$$d_1 = \frac{\log\left(\frac{S_t}{k}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$$
$$d_2 = d_1 - \sigma\sqrt{T-t}$$

and

$$\phi(x) = \mathbb{P}(X \leq x), \quad X \sim \mathcal{N}(0,1)$$

is the c.d.f of the normal law.

This is the famous Black-Scholes formula for the call option [3] . In the case of a put option (the option to sell the underlying) :

$$F(S_t,t) = -S_t \, \phi(-d_1) - ke^{-r(T-t)} \, \phi(-d_2) \tag{19}$$

References

[1] Jean-Philippe Bouchaud and Marc Potters. *Theory of Financial Risk and Derivative Pricing: From Statistical Physics to Risk Management*. 2nd ed. Cambridge University Press, 2003.

[2] C. W. Gardiner. *Handbook of stochastic methods for physics, chemistry and the natural sciences*. Third. Vol. 13. Springer Series in Synergetics. Berlin: Springer-Verlag, 2004. ISBN: 3-540-20882-8.

[3] Fischer Black and Myron Scholes. "The Pricing of Options and Corporate Liabilities". In: *Journal of Political Economy* 81.3 (1973), pp. 637–654.